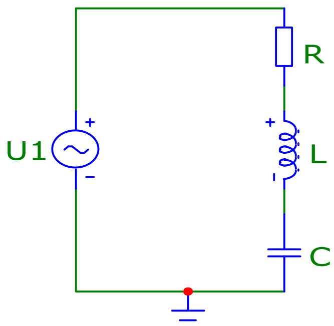
RESONANT CIRCUITS

Motivation:

In physics, resonance is the tendency of a system to oscillate at maximum amplitude at certain frequencies, when small periodic driving forces can produce large amplitude vibrations, because system **accumulates** energy. Well known example of this phenomenon is pendulum, another one is playground swing, in acoustic musical instruments are based on resonance. Sometimes resonance could have destructive effects – we can break a glass by the sound of distinct frequency, even wrong designed buildings and other structures may collapse (e.g. Angers Bridge, see http://en.wikipedia.org/wiki/Angers_Bridge, for a long time Tacoma Narrows Bridge was an another example, before more complicated aeroelastic flutter were discovered).

Let's investigate following RLC circuit ($R = 10\,\Omega$, $L = 1\,\mathrm{H}$, $C = 1\,\mu\mathrm{F}$, if not otherwise specified):



The total impedance of this circuit is

$$\mathbf{Z} = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The magnitude of impedance is

$$|\mathbf{Z}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

It is obvious the magnitude of impedance is frequency dependent, see Figure 1:

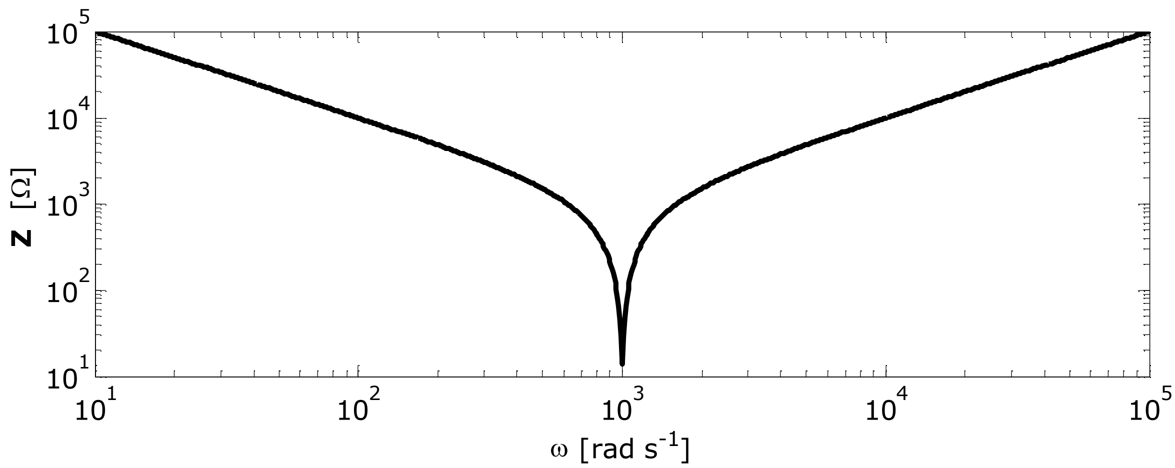


Figure 1 – frequency dependence of impedance magnitude

From the figure is evident, the impedance has minimum in distinct frequency. What is this frequency generally?

$$\frac{\mathrm{d}|\mathbf{Z}|}{\mathrm{d}\omega} = \frac{2\omega L^2 - \frac{2}{\omega^3 C^2}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = 0 \qquad \Longrightarrow \qquad \omega = \frac{1}{\sqrt{LC}}$$

That relationship is called

Thompson's formula

$$\boxed{\omega_r = \frac{1}{\sqrt{LC}}} \qquad [\mathrm{rad\ s^{-1}}]$$

ω_r is resonant frequency of series RLC circuit. When $\omega = \omega_r$, then the impedance

$$\mathbf{Z} = R + j\underbrace{\left(\omega L - \frac{1}{\omega C}\right)}_{=0} = R$$

is real number, $Im\{\mathbf{Z}\} = 0$ is the condition of **voltage resonance**.
It is evident, the current has its maximum value with resonant frequency, see Figure 2, source voltage is 1V.

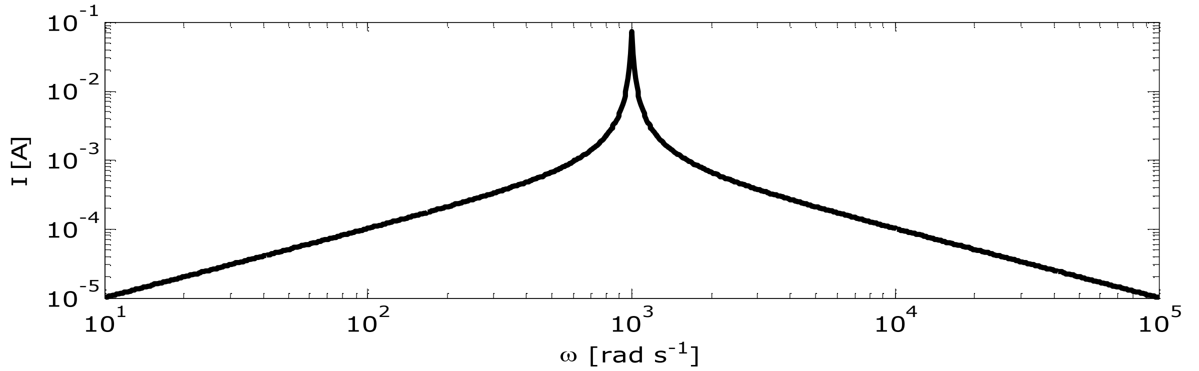


Figure 2 – frequency dependence of passing current

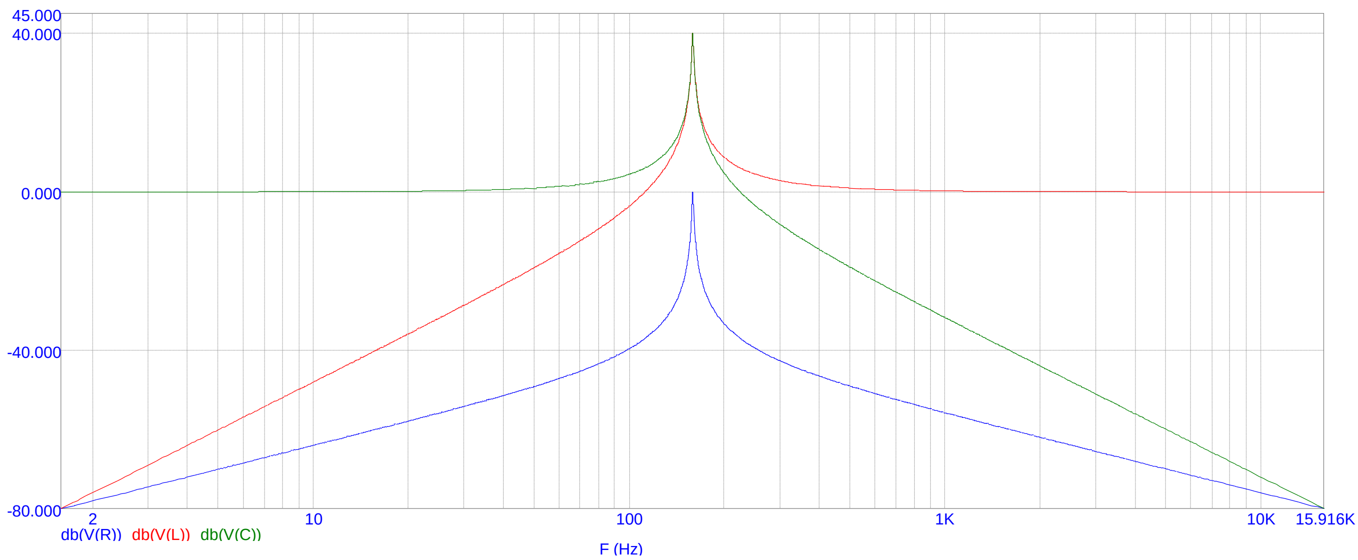


Figure 3 – frequency dependence of voltages

Figure 3 illustrates frequency dependence of voltages
Green is capacitor voltage.
Red is inductor voltage.
Blue is resistor voltage.
Both axes are in logarithmic scale, units on vertical axis are dB, it is $20\log\frac{U_C}{U}$, $20\log\frac{U_L}{U}$ and $20\log\frac{U_R}{U}$. When U = 1 V, then $40\ \mathrm{dB} \approx 100\ \mathrm{V}$.

As voltage may be 100 × greater than source voltage? Let's see the phasor diagram of series RLC in resonance and outside of resonant frequency (see Figure 4) and voltage waveforms (Figure 5).

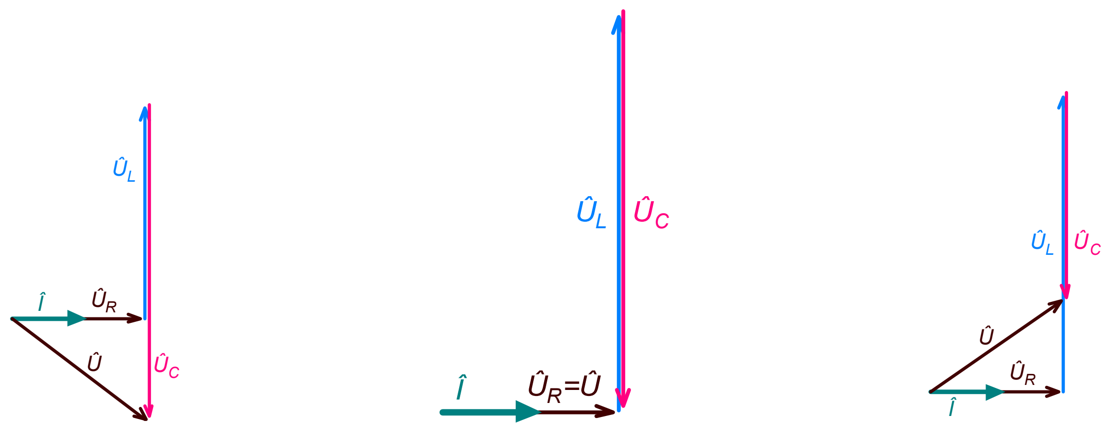


Figure 4 – RLC series circuit phasor diagram $\omega < \omega_r$, $\omega = \omega_r$ and $\omega > \omega_r$

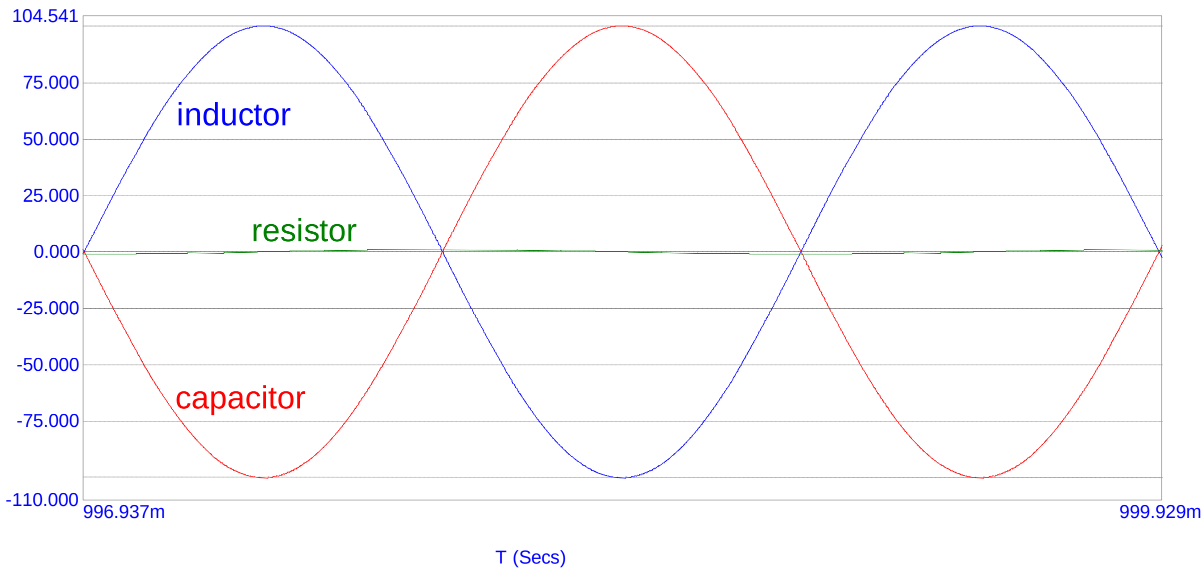


Figure 5 – voltage waveforms across *R*, *L*, and *C* in series RLC resonant circuit, time from approx 0.997 s to 1 s

Voltage across inductor leads the current by 90°, voltage across capacitor lags the current by 90°, so these voltages are shifted by 180° – they have same (absolute) magnitude, but with opposite sign, so they cancels. Then the voltage across resistor is the same voltage as the source voltage.

Figure 5 shows voltage waveforms approximately 1 second after the source was switched on. The reason is, the steady state in the circuit is not reached immediately, but it takes some time before the **maximum voltage** across capacitor / inductor is reached, see Figure 6.

Currently we have not enough mathematical background for exact mathematical proof (it will we shown following semester), so we have to do with simple theoretical concept:

- *Maximum current affected only by resistivity – it determines maximum (steady state) current and hereby voltages*
- *Ideal inductor has zero resistivity – the theory tells us, it is like short circuit – passed by infinite current.* ***But, such state is achieved not until infinitely long period!*** *According to the Faraday's law, there is some voltage only when magnetic flux varies in time – and so electric current varies in time. (Ideal) inductor passed by DC current has no voltage across it. So the current linearly increases in time.*

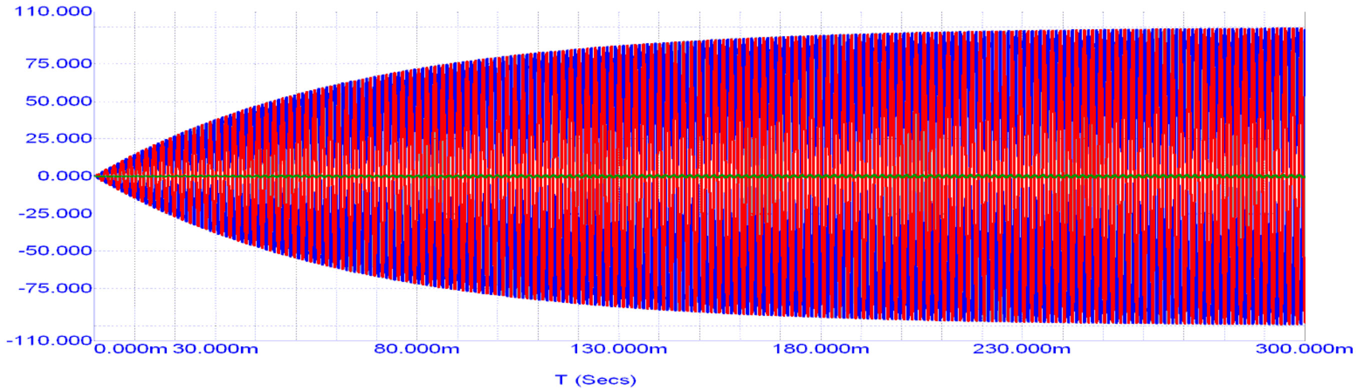


Figure 6 – voltage waveforms across *R*, *L*, and *C* in series RLC resonant circuit, time from 0 s to 0.3 s

Note 1:

*But yet some very brief mathematical description (**don't learn up this now, it will be the case of next semester**):*

Using loop analysis we get an integral-differential equation:

$$Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{C}\int_0^t i(\tau)\mathrm{d}\tau = u(t)$$

Using variation of constants formula we should find roots of quadratic equation

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \qquad\qquad \lambda = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

If $(\frac{R}{2L})^2 < \frac{1}{LC}$ then $\lambda = -\alpha \pm j\omega$, where $\alpha = \frac{R}{2L}$ and the solution is

$$i(t) = K\sin(\omega t + \varphi)e^{-\alpha t} + i_p(t)$$

$i_p(t)$ is steady-state sinusoidal waveform – excited by the source. First part of the solution, exponentially dumped sinusoidal response, is just physical property of the circuit, independent on the excitation. Critical resistivity, when λ is complex conjugated and $i(t)$ has exponentially dumped oscillating response is

$$R_0 = 2\sqrt{\frac{L}{C}}.$$

Only circuits, where $R < R_0$ have oscillating response. Even circuits supplied by the DC source. Of course, oscillations vanishes after short time (commonly accepted is $\frac{3}{\alpha}$).

Here I have to refer to the α variable in the Note 1 – it affect how long does it take to reach steady state. When $t = 5 \cdot \frac{2L}{R}$ s, the waveform reaches approximately 99% of its maximum value. **Lesser *R* means longer time to reach steady state**.

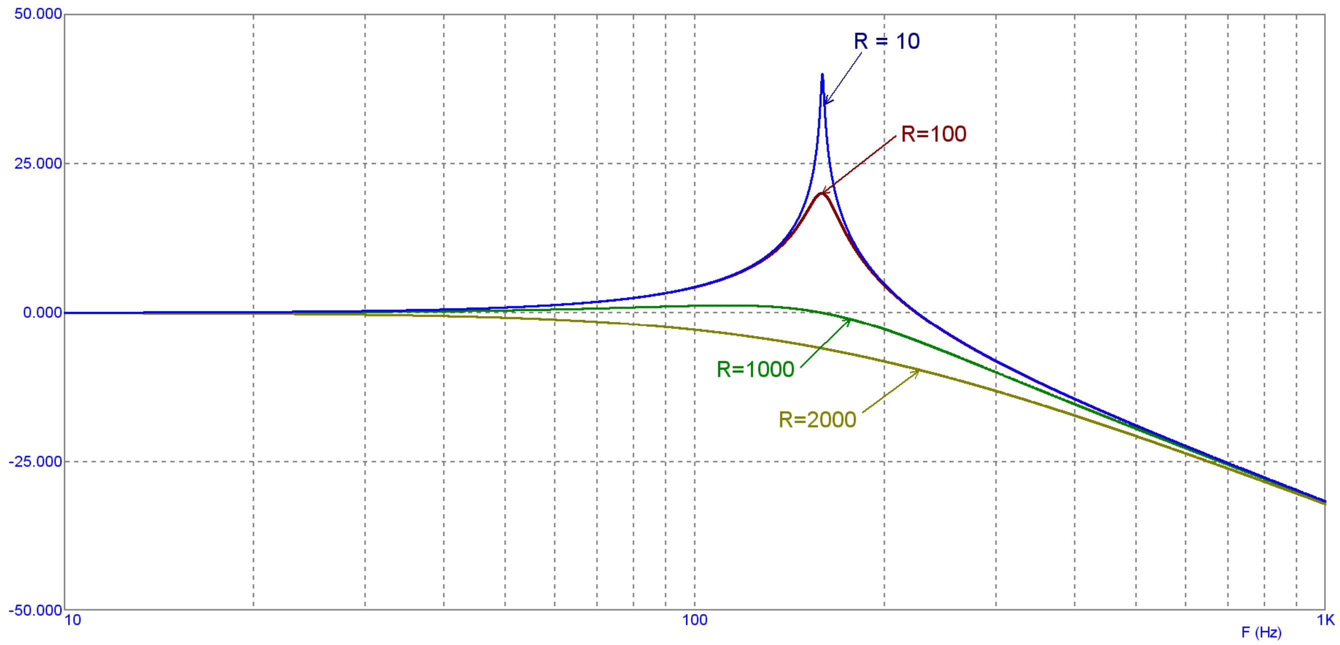


Figure 7 – frequency dependence of voltage across capacitor for different values of R

Figure 7 illustrates frequency response of our series RLC circuit when we change resistivity. It is clear, that our circuit exhibits resonant properties only when $R < 1000$. Below is defined quality factor. When *R* = 1000 Ω, the quality factor of this circuit $Q = \frac{\sqrt{\frac{L}{C}}}{R} = \frac{\sqrt{\frac{1}{10^{-6}}}}{1000} = 1$. Not each series RLC circuit is resonant circuit, only circuits with $Q > 1$ are resonant. The critical resistivity is

$$R_c = \sqrt{\frac{L}{C}}.$$

See critical resistivity R_0 in Note 1 – each series RLC resonant circuit exhibits oscillations when it is connected to DC voltage source, but not every circuit, which exhibits oscillations is resonant (note strong dumping of such oscillations in mentioned interval).

Quality factor

One of the most important uses of resonant circuits is frequency filtering. For example in radio receivers or TV sets pass band filter is required to pass just distinct frequency range respective to the tuned station, whereas other frequencies are cancelled. LC filter based on resonant properties is one option *(even though it is outperformed by crystal filters in today receivers, but this is very illustrative application)*. The higher the ratio of U_L, U_C is, the higher cancelation of undesired frequencies (selectivity) could be achieved and the circuit is "superior". In this way we define the

quality factor – voltage resonance

$$Q = \frac{U_C}{U} = \frac{U_L}{U}$$

Since we may express voltages in resonance as

$$\widehat{\mathbf{U}}_C = \frac{1}{j\omega_r C}\widehat{\mathbf{I}}, \qquad \widehat{\mathbf{U}}_L = j\omega_r L\widehat{\mathbf{I}}, \qquad \widehat{\mathbf{U}} = \widehat{\mathbf{Z}}_r\widehat{\mathbf{I}} = R\widehat{\mathbf{I}},$$

The quality factor results as

$$Q = \frac{1}{\omega_r CR} = \frac{\omega_r L}{R} = \frac{\sqrt{\frac{L}{C}}}{R}$$

Lesser *R* means higher quality factor.

Resonant curve

The Figure 2 illustrates frequency dependence of passing current. For description of properties of resonant circuit the current is expressed in relative form

$$\frac{\mathbf{I}}{\mathbf{I}_r} = \frac{\frac{\mathbf{U}}{\mathbf{Z}_r}}{\frac{\mathbf{U}}{\mathbf{Z}}} = \frac{\mathbf{Z}_r}{\mathbf{Z}} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{1}{1 + j\left(\underbrace{\frac{\omega L}{R}}_{=\frac{1}{Q}|_{\omega=\omega_r}} - \underbrace{\frac{1}{\omega CR}}_{=Q|_{\omega=\omega_r}}\right)}$$

But ω is variable frequency, so to replace statements in the equation by quality factor resonant frequency has to be added there, so we will define relative frequency

$$s = \frac{\omega}{\omega_r}$$

$$\frac{\mathbf{I}}{\mathbf{I}_r} = \frac{1}{1 + j\left(\frac{\omega}{\omega_r}\frac{\omega_r L}{R} - \frac{\omega_r}{\omega}\frac{1}{\omega_r CR}\right)} = \frac{1}{1 + jQ\left(s - \frac{1}{s}\right)}$$

Formally the shape of the curve is the same as on Figure 2, but relative current has values between 0 and 1, in resonance the relative current is equal 1 and relative frequency is also equal to 1, there are 2 important frequencies s_1, s_2 at which the modulus decreases by 3 dB (i.e. $\doteq$ 70.8 % of modulus in resonant frequency) – half power frequencies.

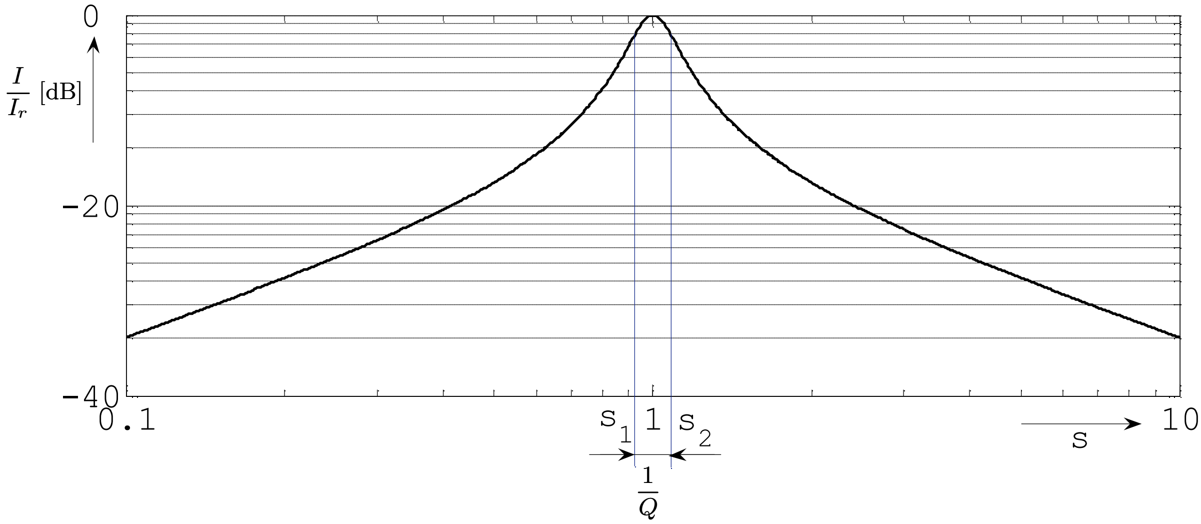


Figure 8 – resonant curve, *Q* = 5

The 3 dB fall is commonly accepted value for determination of the frequency range – the interval of frequencies at which the device passes AC current. Since the vertical axis is logarithmic and according to the rules for drawing of frequnecy characteristics its scale is $20\log\left|\frac{\mathbf{I}}{\mathbf{I}_r}\right|$, the 3 dB fall may be evaluated as follows

$$-3 = 20\log\left|\frac{\mathbf{I}}{\mathbf{I}_r}\right| \qquad \Rightarrow \qquad \left|\frac{\mathbf{I}}{\mathbf{I}_r}\right| = 10^{\frac{-3}{20}} = \frac{1}{\sqrt{2}} = \left|\frac{1}{1\pm j}\right|$$

Then, from resonant curve equation, we get two relations for two distinct frequencies

$$Q\left(s_1 - \frac{1}{s_1}\right) = -1 \qquad \text{and} \qquad Q\left(s_2 - \frac{1}{s_2}\right) = 1$$

From here it follows, that

$$s_1 = \frac{1}{s_2} \qquad \text{(symmetry)}$$

$$\frac{1}{Q} = s_2 - s_1 \qquad \text{(bandwidth).}$$

Figure 9 shows three different resonant curves for three different quality factors – *Q* = 10 (the green one), *Q* = 50 (the red) and *Q* = 100 (black). Higher quality factor implicate narrower peak and higher dumping at other than resonant frequencies.

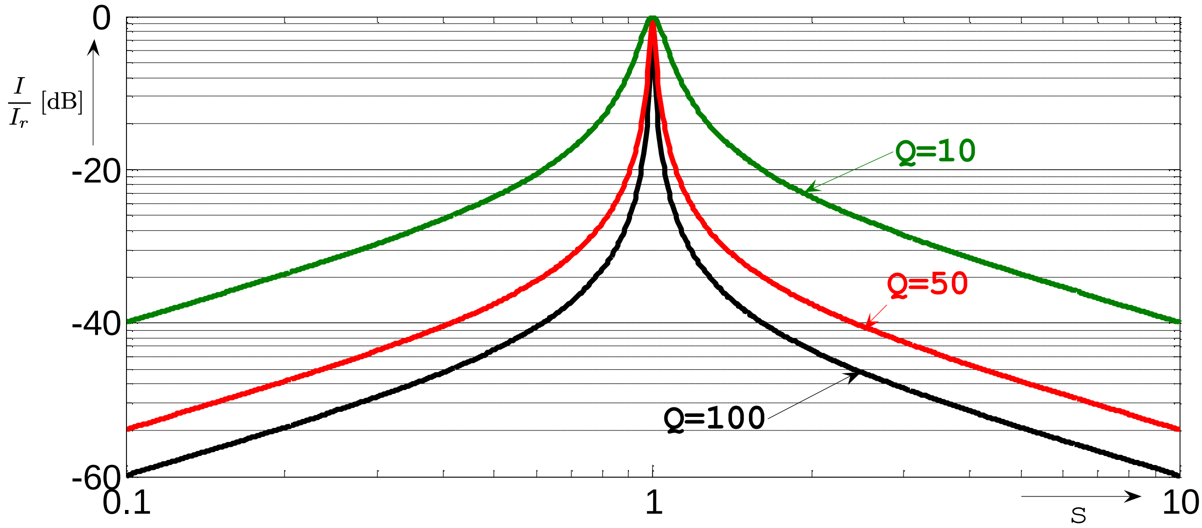


Figure 9 – resonant curve, Q = 100, Q = 50 and Q = 10

But remember, this is current, the important for us is voltage (see Figure 3).

Parallel RLC circuit – current resonance

In ideal parallel RLC circuit each circuit element has same voltage, but currents may be different.

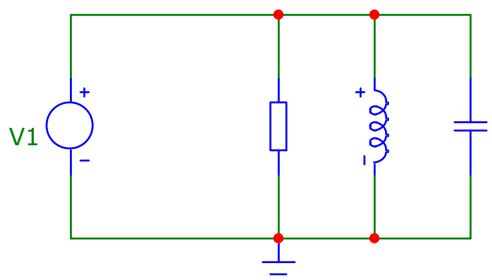
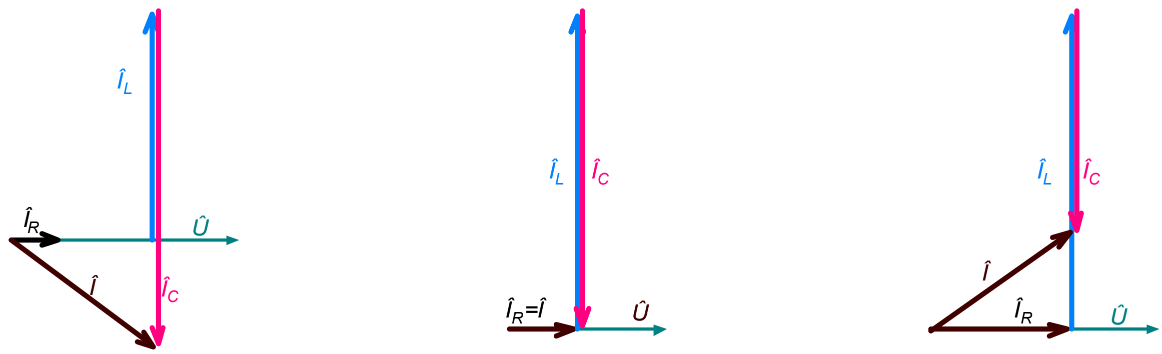


Figure 10 – parallel RLC circuit



In resonance, $|I_L| = |I_C|$, but they are phase shifted by 180° (opposite orientation on phasor diagram). Since

$$\mathbf{I}_L = \mathbf{Y}_L\mathbf{U} = \frac{1}{j\omega L}\mathbf{U}, \quad \mathbf{I}_C = \mathbf{Y}_C\mathbf{U} = j\omega C\mathbf{U}$$

susceptances of capacitor and inductor in resonance must be equal

$$\frac{1}{\omega_r L} = \omega_r C$$

and resonant frequency is

$$\omega_r = \frac{1}{\sqrt{LC}}.$$

Then the total addmitance of the circuit

$$\mathbf{Y} = G + j\underbrace{\left(\omega C - \frac{1}{\omega L}\right)}_{=0} = G$$

is real number. Here, the quality factor is ratio of current passing inductor / capacitor, and current loaded from source.

$$Q = \frac{I_C}{I} = \frac{I_L}{I}$$

By substitution of I_L, I_C (see forth above) and $I_r = GU$ result relationship

$$Q = \frac{\omega_r C}{G} = \frac{1}{\omega_r L G}$$

Opposite to **voltage resonance**, where in high quality resonant circuits **resistivity is** quite **low** (zero in ideal), in **current resonance** high quality resonant circuit has low conductivity and so **high resistivity** (infinite in ideal case).

Now we will consider following values of circuit elements: $R = 10\,\mathrm{k\Omega}$, $L = 1\,\mathrm{H}$, $C = 1\,\mu\mathrm{F}$.

Figure 11 shows the waveforms of currents in the parallel RLC circuit if it is supplied from the voltage source, *U* = 1V. The waveforms do not change in time. Since the current passing inductor is continuous and before we connect the source this current was zero and its waveform is sinusoidal, it has superimposed DC current. There is no energy accumulation effect.

Figure 12 shows the same circuit when it is supplied from the current source, *I* = 1 mA. The magnitude of currents increases before it reaches steady state – the capacitor and inductor interchanges mutually energy and current source just supplement losses in resistor (in the steady state). Compare maximum charge stored in capacitor and charge delivered by the current. It is in contrast with voltage excitement. Ideal parallel RLC circuit exhibits resonant properties when it is supplied from current source.

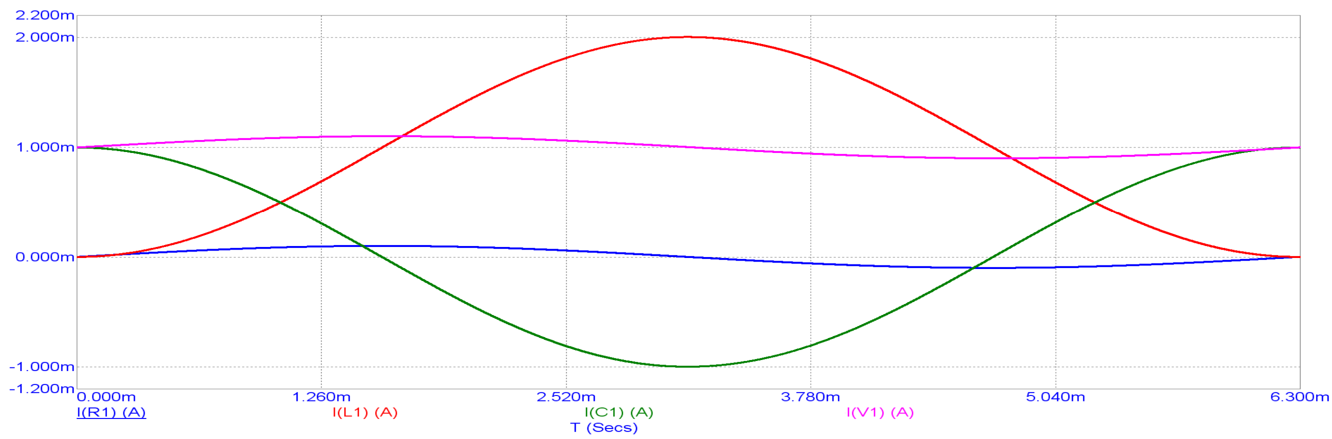


Figure 11 – parallel RLC circuit, voltage excitation

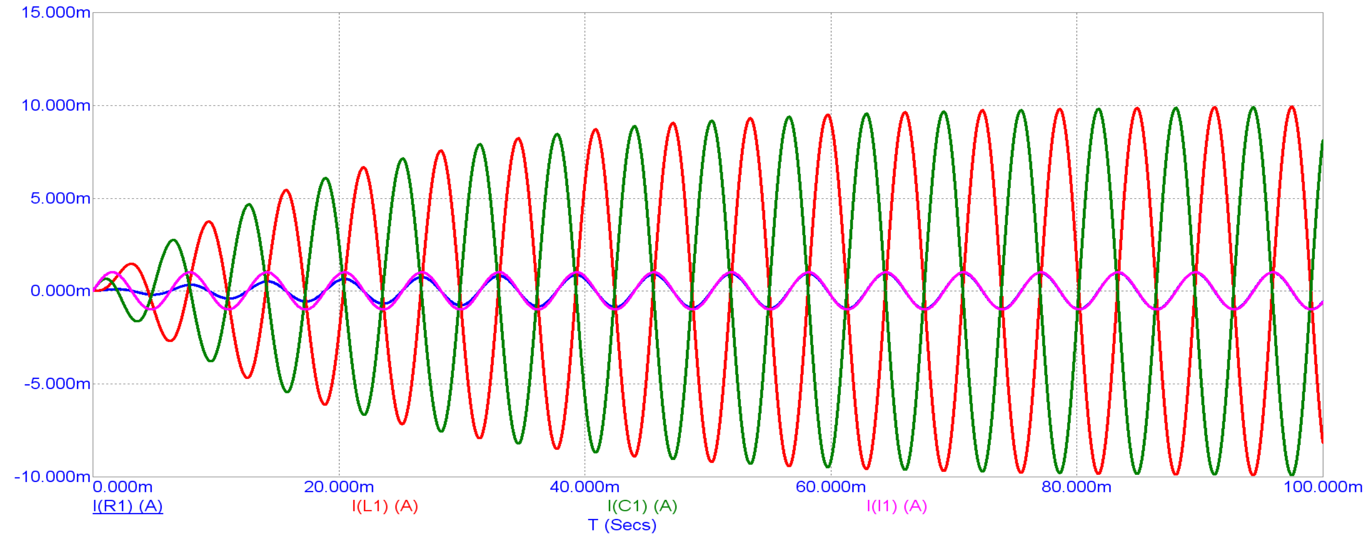


Figure 12 – parallel RLC circuit, current excitation

But, **it is not possible to realize such circuit practically** by coils and condensers in usual conditions, so it has only theoretical meaning. **Actual parallel resonant circuit** is shown on Figure 13.

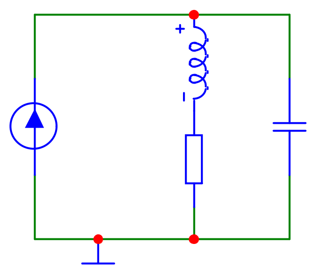


Figure 13 – actual parallel resonant circuit

In resonance, the phase shift between currents is less than 180°, see phasor diagram on Figure 14. Supplying current and total voltage has to be in phase, because total admittance (and so impedance) in resonance has to be real.

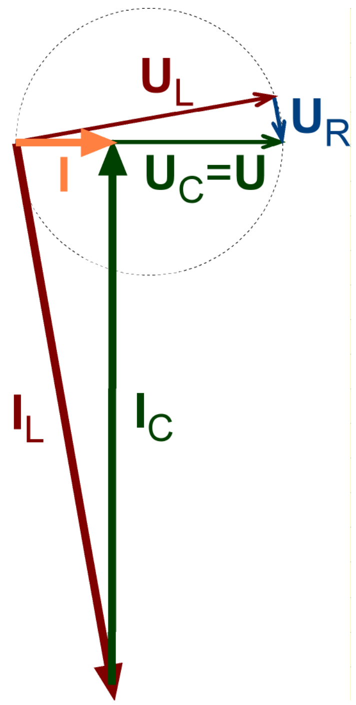


Figure 14 – phasor diagram of actual parallel resonant circuit

The admittance of this circuit is

$$\mathbf{Y} = j\omega C + \frac{1}{R + j\omega L} = \frac{R}{R^2 + \omega^2 L^2} + j\underbrace{\left(\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)}_{=0}$$

Hence the **resonant frequency**

$$\omega_r = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

is different from Thomson formula. And it is general result – circuit is in resonance when imaginary part of its impedance (voltage resonance) and/or admittance (current resonance) is zero, the frequency could be different from Thomson formula, and circuit may have even more resonant frequencies.

The currents have different magnitude, so which one we should use to determine the quality factor? Moreover, in actual circuit could be other loss elements (internal resistances, leakage …), so quality factor definitions mentioned above introduce inaccuracies. Universal definition is

Energetic quality factor definition

$$Q = 2\pi \frac{\text{energy stored in the resonant circuit}}{\text{energy dissipated into heat within one period}}$$

Here, the instantaneous stored energy may be evaluated

$$W_s(t) = \frac{1}{2} L i_L^2(t) + \frac{1}{2} C u_C^2(t)$$

Practical example of resonant circuits is band pass and band stop filters; another example is power factor compensation, see previous lecture.