Fourier transform

EO2 – Lecture 2

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We already know complex form of Fourier series

\[ f(t) = \sum_{k=-\infty}^{\infty} A_k e^{jk\omega_0 t} \]

\[ A_k = \frac{1}{T} \int_{0}^{T} f(t) e^{-jk\omega_0 t} dt \]

- Series frequency spectra is **discrete**
- Circuits in the sinusoidal steady state must be solved **step by step** (very laborious)
- We can expand only **periodic waveforms**

? What if we connect to the circuit single pulse source?
? How to find output voltage by one computation, not laborious step by step calculation, term by term?
What if we will extend period of rectangular pulse?
Or narrow rectangular pulse?

\[
A_k = \frac{1}{T} \int_{-t_0/2}^{t_0/2} U_0 e^{-j k \omega_0 t} \, dt = \frac{U_0}{T} \left[ \frac{e^{-j k \omega_0 t_0}}{-j k \omega_0} \right]^{t_0/2}_{-t_0/2} = \\
= \frac{U_0}{T} \frac{\cos(-k \omega_0 t_0/2) + j \sin(-k \omega_0 t_0/2) - \cos(k \omega_0 t_0/2) - j \sin(k \omega_0 t_0/2)}{-j k \omega_0} \\
= \frac{U_0}{2} \frac{2 \sin(k \omega_0 t_0/2)}{k \omega_0} = \frac{U_0 t_0}{T} \frac{\sin(k \omega_0 t_0/2)}{k \omega_0 t_0/2} = \frac{S}{T} S_{\delta} \left( k \omega_0 t_0/2 \right)
\]
Both when we narrow the pulse, or when we extend the period, the envelope curve is \( \frac{\sin x}{x} \) function.

The larger the ratio of period \( T \) and pulse width \( t_0 \) is, the more spectral terms falls in one period of the \( \frac{\sin x}{x} \) function.

**Narrowing** – constant \( T \), lesser \( t_0 \), \( k \) represents still the same frequency (frequency) distance of neighbouring terms is the same – \( \frac{\sin x}{x} \) function extends across towards \( \infty \)

**Widening** – constant \( t_0 \), \( T \) extends towards \( \infty \), \( k \) represents still less frequencies (frequency) of neighbouring terms falls up to zero – \( \frac{\sin x}{x} \) function stay at place, but magnitude plunge!
Period extension towards $\infty$

$$ T \to \infty \quad \Rightarrow \quad \omega_0 = \frac{2\pi}{T} \to 0, \quad \Delta \omega_0 = \omega_0 \to 0 $$

- Initially discrete frequency becomes **continuous**
- **Magnitude** of frequency spectra (primarily distinct harmonics) **tends to 0**

**Coefficients** (harmonics) have to be multiplied by period $T$ (or $\to 0$ !)

$$ T A_k = \int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t} \, dt \quad \frac{T}{2} \to \infty $$

$$ F(j\omega) = \lim_{T \to \infty} T A_k = \lim_{T \to \infty} \int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t} \, dt = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt $$

**Direct Fourier transform**

$$ \mathcal{F} \{ f(t) \} = F(j\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t} \, dt $$
The waveform is qualified by series

\[ f(t) = \sum_{k=-\infty}^{\infty} (TA_k)e^{j\omega_0 t} \cdot \frac{1}{T} = \sum_{k=-\infty}^{\infty} (TA_k)e^{j\omega_0 t} \cdot \frac{1}{2\pi \omega_0} = \]

\[ = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (TA_k)e^{j\omega_0 t} \cdot \omega_0 = |\Delta \omega = \omega_0| = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (TA_k)e^{j\omega_0 t} \Delta \omega \]

If \( T \to \infty \), then

\[ \Delta \omega \to d\omega, \quad \sum \to \int \]

Inverse Fourier transform

\[ \mathcal{F}^{-1} \{F(j\omega)\} = f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t} d\omega \]

So far is possible, we don’t use definition integral directly, but we try to use transform properties and known transforms
Function has to satisfy \textbf{Dirichlet’s conditions} (Fourier series !)

Function has to be \textbf{absolutely integrable}

\[ \int_{-\infty}^{\infty} |f(t)| \, dt < \infty \]

\textbullet~considerably restrictive condition, when transform is not applicable to so common waveform, such as unit step function (DC voltage / current)

\textbullet~It satisfies condition

\[ \lim_{t \to +\infty} f(t) = 0 \]

\textbullet~Transform is essential tool of description of frequency properties of discrete systems (sound and image digital processing – CD, SACD and DVD players, home cinemas, ...)
The transform is a analysis tool of frequency spectra of waveforms – the maxima of $F(j\omega)$ functions denotes harmonics of frequency spectra.

$$u(t) = (\sin(2\pi t) + 2\sin(6\pi t)) e^{-\pi t^2}$$

Fourier transform

$$U(j\omega) = \frac{-j}{2} e^{\frac{1}{4} \frac{\omega^2 + 12\omega \pi + 36\pi^2}{\pi}} (2 e^{6\omega} + e^{4\omega + 8\pi} - e^{2\omega + 8\pi} - 2)$$

Dumping is essential to satisfy conditions of existence of the transform

Function is „stretched“ ⇒ it has limited frequency resolution
## BASIC PROPERTIES

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<tr>
<th>property</th>
<th>time domain</th>
<th>frequency domain</th>
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<tr>
<td>linearity</td>
<td>$f(t) = af_1(t) + bf_2(t)$</td>
<td>$F(j\omega) = aF_1(j\omega) + bF_2(j\omega)$</td>
</tr>
<tr>
<td>Time shifting</td>
<td>$f(t) = f_1(t - t_0)$</td>
<td>$F(j\omega) = F_1(j\omega)e^{-j\omega t_0}$</td>
</tr>
<tr>
<td>Frequency shifting (modulation)</td>
<td>$f(t) = f_1(t)e^{j\omega_0 t}$</td>
<td>$F(j\omega) = F_1(j(\omega - \omega_0))$</td>
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<tr>
<td>Differentiation</td>
<td>$f(t) = \frac{d^n f_1(t)}{dt^n}$</td>
<td>$F(j\omega) = (j\omega)^n F_1(j\omega)$</td>
</tr>
<tr>
<td>Integration</td>
<td>$f(t) = \int_{-\infty}^{t} f_1(t) , dt$</td>
<td>$F(j\omega) = \frac{F_1(j\omega)}{j\omega}$</td>
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<tr>
<td>Scaling</td>
<td>$f(t) = f_1(at)$</td>
<td>$F(j\omega) = \frac{1}{</td>
</tr>
<tr>
<td>Exponential pulse transform</td>
<td>$f(t) = e^{-at}$</td>
<td>$F(j\omega) = \frac{1}{j\omega + a}$</td>
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</table>
Find the Fourier series of rectangular waveform on the figure. 

\( U_m = 2 \, \text{V}, \, T = 0.1 \, \text{s}, \, t_0 = 0.025 \, \text{s} \)

\[
U_k = \frac{1}{T} \int_{-t_0/2}^{t_0/2} U_m e^{-j k \omega_0 t} \, dt = \frac{U_m}{T} \left[ \frac{e^{-j k \omega_0 t}}{-j k \omega_0} \right]_{-t_0/2}^{t_0/2} = \frac{U_m t_0}{T} \frac{\sin(k \omega_0 t_0)}{k \omega_0 t_0} = \frac{2}{k \pi} \sin \frac{k \pi}{4}
\]

\[ \omega_0 = \frac{2 \pi}{T} = \frac{2 \pi}{0.1} = 20 \pi \quad \Rightarrow \quad k = 16 \quad \Rightarrow \quad \omega = 16 \cdot 20 \pi \approx 1005.31 \, \text{s}^{-1} \]
Find the Fourier transform of the rectangular pulse on the figure. Compare result with Fourier series above.

\[ U_m = 2 \text{ V}, \ t_0 = 0.025 \text{ s} \]

\[ F(j\omega) = \int_{-t_0/2}^{t_0/2} U_m e^{-j\omega t} \, dt = U_m \left[ \frac{e^{-j\omega t_0}}{-j\omega} \right]_{-t_0/2}^{t_0/2} = U_m \frac{2 \sin \omega t_0/2}{\omega} = U_m t_0 \frac{\sin \omega t_0/2}{\omega t_0/2} = 0.05 \frac{\sin 0.0125\omega}{0.0125\omega} \]

Magnitude of Fourier transform changes with the length of the pulse!
Find the Fourier transform of the pulse on the figure.

\[ U_1 = 1 \text{ V}, \quad U_2 = 2 \text{ V}, \quad t_0 = 0.05 \text{ s} \]

- The waveform may be considered as superposition of two distinct waveforms
  - rectangular, \( U_m = 1 \text{ V} \)
  - saw tooth, \( U_m = 1 \text{ V} \)

- We know the transform of rectangular waveform from previous example, but we have to modify it – the magnitude is different, as well as time \( t_0 \) and it is shifted in time
  
  - applied properties are \textbf{scaling}, \( a = 0.5 \) and \textbf{time shifting} within \(-0.025 \text{ s}\)

\[
F_1(j\omega) = \frac{1}{2} \cdot 2 \cdot 0.05 \cdot \frac{\sin 2 \cdot 0.0125\omega}{2 \cdot 0.0125\omega} \cdot e^{-j\omega 0.025} = 0.05 \cdot \frac{\sin 0.025\omega}{0.025\omega} e^{-j\omega 0.025}
\]

\[
F_2(j\omega) = \int_0^{t_0} \frac{U_m}{t_0} t e^{-j\omega t} \, dt = \left| \begin{array}{c} u = t \\ v' = e^{-j\omega t} \\ u' = 1 \\ v = \frac{e^{-j\omega t}}{-j\omega} \end{array} \right| = \frac{U_m}{t_0} \left[ \frac{e^{-j\omega t}}{-j\omega} - \frac{e^{-j\omega t}}{(-j\omega)^2} \right]_0^{t_0} = \frac{U_m}{t_0 \omega^2} \left( e^{-j\omega t_0} (1 + j\omega t_0) - 1 \right)
\]
In contrast to Fourier series (where we have to compute with each harmonic separately), we may find waveforms of circuit variables with non-sinusoidal excitation likewise using sinusoidal steady state analysis

1. Find Fourier transform of excitation pulse
   \[ x_1(t) \rightarrow X_1(j\omega) \]
2. Using transfer function (same, as in sinusoidal steady state), find transform of output voltage
   \[ X_2(j\omega) = P(j\omega)X_1(j\omega) \]
3. Using inverse transform we find output voltage waveform
   \[ X_2(j\omega) \rightarrow x_2(t) \]

When we know input and output voltage waveforms, we can find frequency response of the circuit

1. Find Fourier transform of input waveform
   \[ x_1(t) \rightarrow X_1(j\omega) \]
2. Find Fourier transform of output waveform
   \[ x_2(t) \rightarrow X_2(j\omega) \]
3. Find transfer function of the circuit
   \[ P(j\omega) = \frac{X_2(j\omega)}{X_1(j\omega)} \]
Integrating network in the figure is excited by rectangular pulse on the figure. Find waveform of the output voltage.

1. \[ U_1(j\omega) = \int_0^{t_0} U_m e^{-j\omega t} dt = U_m \left[ \frac{e^{-j\omega t}}{-j\omega} \right]_0^{t_0} = \frac{U_m}{-j\omega} \left[ e^{-j\omega t_0} - 1 \right] \]

2. \[ P(j\omega) = \frac{1}{1 + j\omega RC} \]

3. \[ U_2(j\omega) = \frac{1}{1 + j\omega RC} \cdot \frac{U_m}{-j\omega} \left[ e^{-j\omega t_0} - 1 \right] \]

Now, we should „just“ find inverse transform...

1. We know, \( \frac{1}{j\omega} \) is transform of the integral \( \mathcal{F} \left\{ \int_0^{t_0} u'_2(t) dt \right\} = \frac{1}{j\omega} \cdot U_m \frac{1 - e^{-j\omega t_0}}{1 + j\omega RC} \)
   - first, we find function \( u'_2(t) \) and then we will integrate it

2. The transform \( U_m \frac{1 - e^{-j\omega t_0}}{1 + j\omega RC} \) is superposition of two different functions, \( e^{-j\omega t_0} \) is the time delay \( t_0 \)

3. \[ \mathcal{F}^{-1} \left\{ \frac{1}{1 + j\omega RC} \right\} = \frac{1}{RC} e^{-\frac{1}{RC}t} \Rightarrow u'_2(t) = \frac{U_m}{RC} \left( e^{-\frac{1}{RC}t} \cdot 1(t) - e^{-\frac{1}{RC}(t-t_0)} \cdot 1(t-t_0) \right) \]

4. \[ \int_0^{t} u'_2(t) dt = \frac{U_m}{RC} \left( \int_0^{t} e^{-\frac{1}{RC}t} dt - \int_0^{t} e^{-\frac{1}{RC}(t-t_0)} dt \right) = U_m \left[ (1 - e^{-\frac{1}{RC}t})1(t) - (1 - e^{-\frac{1}{RC}t_0})1(t-t_0) \right] \]

Note. – it is possible find it more easily – partial fractions, Laplace transform...
To the input of the circuit we connected the waveform \( u_1(t) = 10e^{-500t} \) V.

On the output we measured the waveform \( u_2(t) = 6.6(e^{-500t} - e^{-2000t}) \) V.

Find transfer function of the circuit. Find suitable circuit diagram.

\[
U_1(j\omega) = \frac{10}{j\omega + 500}
\]

\[
U_2(j\omega) = \frac{6.6}{j\omega + 500} - \frac{6.6}{j\omega + 2000} = \frac{6.6 \cdot 2000 - 6.6 \cdot 500}{(j\omega + 500)(j\omega + 2000)} = \frac{10000}{(j\omega + 500)(j\omega + 2000)}
\]

\[
P(j\omega) = \frac{U_2(j\omega)}{U_1(j\omega)} = \frac{10000}{(j\omega+500)(j\omega+2000)} \cdot \frac{\frac{10}{j\omega+500}}{\frac{10}{j\omega+2000}} = \frac{1000}{j\omega + 2000}
\]