Circuit equations in time domain and frequency domain

EO2 – Lecture 5
Pavel Máša
In last semester we studied circuit equations in DC and AC circuits.

What is the same and what is different when we will write circuit equations in time domain or in operational form, or in DC or AC circuits?

Circuit equations, regardless of used mathematical apparatus, are always mathematical formulation of Kirchhoff’s laws:

**MESH (LOOP) ANALYSIS – KVL**

\[ \sum_{k} U_k = 0 \]

voltage across R, L, C is qualified by means of current

**NODAL ANALYSIS – KCL**

\[ \sum_{k} I_k = 0 \]

current, passing R, L, C is qualified by means of voltage
## Relationship Between Voltage and Current

<table>
<thead>
<tr>
<th>Circuit element</th>
<th>Time domain</th>
<th>DC</th>
<th>AC / Fourier</th>
<th>Laplace</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R</strong></td>
<td></td>
<td>$u_R(t) = Ri_R(t)$</td>
<td>$U_R = RI_R$</td>
<td>$U_R(p) = RI_R(p)$</td>
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<tr>
<td></td>
<td></td>
<td>$i_R(t) = Gu_R(t)$</td>
<td>$I_R = GU_R$</td>
<td>$I_R(p) = GU_R(p)$</td>
</tr>
<tr>
<td><strong>L</strong></td>
<td>$u_L(t) = L \frac{di_L(t)}{dt}$</td>
<td>$U_L = 0$</td>
<td>$U_L = j\omega LI_L$</td>
<td>$U_L(p) = pL I_L(p) - LI_L(0_+)$</td>
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<tr>
<td></td>
<td>$i_L(t) = \frac{1}{L} \int_0^t u_L(\tau) , d\tau + i_L(0_+)$</td>
<td>$I_L = \text{any short}$</td>
<td>$I_L = \frac{1}{j\omega L} U_L$</td>
<td>$I_L(p) = \frac{1}{pL} U_L(p) + \frac{i_L(0_+)}{p}$</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>$u_C(t) = \frac{1}{C} \int_0^t i_C(\tau) , d\tau + u_C(0_+)$</td>
<td>$U_C = \text{any}$</td>
<td>$U_C = \frac{1}{j\omega C} I_C$</td>
<td>$U_C(p) = \frac{1}{pC} I_C(p) + \frac{u_C(0_+)}{p}$</td>
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<td>$i_C(t) = C \frac{du_C(t)}{dt}$</td>
<td>$I_C = 0$</td>
<td>$I_C = j\omega C U_C$</td>
<td>$I_C(p) = pC U_C(p) - CU_C(0_+)$</td>
</tr>
</tbody>
</table>

### Usage
- **Transient analysis**
  
  *usually DC/AC analysis has to be also proceeded*

- Steady state with DC source
  
  *AC – steady state with sine wave source*
  
  *Fourier series – steady state with periodical excitation*
  
  *transform – pulse excitation*

- **General** application contains both steady and transient components
Energetic initial conditions
– since energy is continuous, energetic circuit variables are also continuous

<table>
<thead>
<tr>
<th>Capacitor</th>
<th>Inductor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = Cu$</td>
<td>$\Phi_C = Li$</td>
</tr>
<tr>
<td>$i = \frac{dq}{dt}$</td>
<td>$u(t) = \frac{d\Phi_C}{dt}$</td>
</tr>
<tr>
<td>$i_C(t) = C \frac{du_C(t)}{dt}$</td>
<td>$u(t)$</td>
</tr>
</tbody>
</table>

- must be $u_C(0_-) = u_C(0_+)$
- may be $i_C(0_-) \neq i_C(0_+)$

- may be $u_L(0_-) \neq u_L(0_+)$
- must be $i_L(0_-) = i_L(0_+)$

- History of capacitor describes charge stored in the capacitor $\Rightarrow$ voltage
- History of inductor describes magnetic flux passing the inductor $\Rightarrow$ electric current
**Mesh analysis**

- Current in the loop, where we currently write the equation has always positive sign
- Other currents passing circuit element, where we currently evaluate voltage, have positive voltage, when they have same orientation, negative, when they have opposite orientation
- Voltage sources have positive sign, when the current in the loop, where we currently write the equation, flows into positive source terminal, and negative, when it flows into negative terminal
- We could not write an equation in loops, passing current sources, but we have to count current sources in closed loops

**Nodal voltages**

- Voltage in the node, where we write an equation, has always positive sign
- Voltages in adjacent nodes, with the exception of voltage sources, have always negative sign *(we suppose, all currents leaves the node, actual orientation results from solution of system of equations; circuit element voltage, is the difference of electric potentials)*
- Voltage sources in adjacent nodes have negative sign, when they are connected by positive terminal to the adjacent node (we subtract them), positive, if they are connected by negative terminal
- Current of the current source has positive sign, if it leaves the node, negative, it runs into the node
- When the voltage source is not connected by any terminal with reference node (ground), then we refer it as floating source – we write just one equation for both nodes (where the floating source is connected).
Reduction of number of equations

DC
\[ U = RI \]

AC
\[ U = R I + j \omega L I = (R + j \omega L) I \]

Time domain
\[ u(t) = R \dot{i}(t) + L \frac{di(t)}{dt} \quad \text{not possible} \]

Laplace
\[ U(p) = R I(p) + pL I(p) - \dot{L}i_L(0) = (R + pL) I(p) - \dot{L}i_L(0) \]

\[ \frac{U_a - U_1}{R_1} + \frac{U_A}{R_2} + \frac{U_A + U_2}{R_3} = 0 \]

AC
\[ \frac{U_a - U_1}{R_1} + \frac{U_A}{R_2 + j \omega L} + \frac{U_A + U_2}{R_3} = 0 \]

Time domain
\[ \frac{u_a(t) - u_1(t)}{R_1} + \frac{u_A(t) - u_B(t)}{R_2} + \frac{u_A(t) + u_2(t)}{R_3} = 0 \]
\[ \frac{u_b(t) - u_a(t)}{R_2} + \frac{1}{L} \int_0^t u_B(\tau) \, d\tau + i_L(0) = 0 \]

Laplace
\[ \frac{U_a(p) - U_1(p)}{R_1} + \frac{U_A(p) + \dot{L}i_L(0)}{R_2 + pL} + \frac{U_A(p) + U_2(p)}{R_3} = 0 \]
**COUPLED INDUCTORS (TRANSFORMER)**

**Time domain – mesh analysis**

\[
\begin{align*}
    u_1(t) &= L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt} \\
    u_2(t) &= L_2 \frac{di_2(t)}{dt} + M \frac{di_1(t)}{dt}
\end{align*}
\]

**Laplace transform – mesh analysis**

\[
\begin{align*}
    U_1(p) &= pL_1 I_1(p) - L_1 i_1(0+) + pM I_2(p) - M i_2(0+) \\
    U_2(p) &= pL_2 I_2(p) - L_2 i_2(0+) + pM I_1(p) - M i_1(0+)
\end{align*}
\]

– nodal voltages

**Nodal voltages**

\[
\begin{align*}
    I_2(p) &= \frac{U_2(p) + L_2 i_2(0+) - pM I_1(p) + M i_1(0+)}{pL_2} \\
    U_1(p) &= pL_1 I_1(p) - L_1 i_1(0+) + pM \frac{U_2(p) + L_2 i_2(0+) - pM I_1(p) + M i_1(0+)}{pL_2} - M i_2(0+) \\
    I_1(p) &= \frac{1}{p} \frac{L_2}{L_1 L_2 - M^2} U_1(p) - \frac{1}{p} \frac{M}{L_1 L_2 - M^2} U_2(p) - \frac{i_1(0+)}{p} = \frac{1}{p} \Gamma_1 U_1(p) + \frac{1}{p} \Gamma M U_2(p) - \frac{i_1(0+)}{p} \\
    I_2(p) &= \frac{1}{p} \Gamma M U_1(p) + \frac{1}{p} \Gamma_2 U_2(p) - \frac{i_2(0+)}{p}
\end{align*}
\]
Example:

\[ u_C(0) \downarrow \quad C \quad \downarrow i_L(0) \]

**DC:**
Mesh analysis – 0 equations, voltages across \( R_1 \) and \( R_2 \) is evaluated by Ohm’s law

**AC:**
\[ j\omega L \hat{i}_1 + R_2 \hat{i}_1 + \frac{1}{j\omega C} (\hat{i}_1 - \hat{i}) = 0 \quad \Rightarrow \quad \hat{i}_1 = \frac{\hat{i}}{j(\omega)^2 LC + j\omega R_2 C + 1} \]

**Time domain:**
\[ L \frac{d\hat{i}_1(t)}{dt} + R_2 \hat{i}_1(t) + \frac{1}{C} \int_0^t [\hat{i}_1(\tau) - \hat{i}(\tau)] \, d\tau - u_C(0) = 0 \]

Solution is obtained in three steps:
1. Initial condition \( u_C(0) \) – DC or AC analysis has to be proceeded according to the nature of exciting source before the change
2. Solution of integral-differential equation – solution of the transient
3. To find steady state in the circuit after transient is over we have to proceed DC or AC analysis again

**Usage – transients** – describing currents or voltages in circuit after
- Source was connected
- Source was disconnected
- Circuit layout has been changed
- Some circuit element parameters (resistivity, capacitance, inductance)

**DC or AC analysis always has to be proceeded**
Laplace: \[ pL I_1(p) - Li_L(0) + R_2 I_1(p) + \frac{1}{pC} [I_1(p) - I(p)] - \frac{u_C(0)}{p} = 0 \]

\[ \Rightarrow \quad I_1(p) = \frac{I(p) + C u_C(0) + pL C i_L(0)}{p^2 L C + p R_2 C + 1} \]

The solution is obtained in two steps:

1. Initial condition \( u_C(0) \) DC or AC analysis has to be proceeded according to the nature of exciting source before the change (*same as time domain*)
2. Solve inverse transform \( I_1(p) \) – it contains both transient and steady state after transient dies away

- To find initial conditions, we have to proceed DC or AC analysis as well
- Inverse Laplace transform always corresponds to transient when source \( I(p) \) is connected to the circuit (if it is present in the equation), or source is disconnected (then the steady state is „hidden“ in the initial conditions)
  - The transient is integral part of Laplace transform solution, the result of time interval restriction
    - each source is zero when \( t < 0 \), because \( u(t) = u'(t) 1(t) \), \( i(t) = i'(t) 1(t) \)
- The solution of operational circuit equations contains also steady state after transient dies away
- Although operational solution is formally similar to the AC analysis, it is different – „more complete“ solution
• Now assume following values: \( R_2 = 1 \, \text{k}\Omega, \, L = 1\, \text{H}, \, C = 1 \, \mu\text{F}, \, R_1 = 2 \, \text{k}\Omega \)
• DC – Circuit supplied by current source \( I = 10 \, \text{mA} \)

\[
U_2 = R_2 I = 1000 \cdot 0.01 = 10 \text{ V} \quad U_1 = R_1 I = 2000 \cdot 0.01 = 20 \text{ V}
\]

• AC – circuit supplied from sinusoidal current source \( i(t) = 10 \sin(1000t) \, \text{mA} \)

\[
i_1(t) = \frac{\hat{I}}{(j \omega)^2 LC + j \omega R_2 C + 1} = \frac{0.01}{(j \cdot 1000)^2 \cdot 1 \cdot 10^{-6} + j \cdot 1000 \cdot 10^{-6} + 1} = 10e^{-\pi j} \, \text{mA}
\]

• Time domain – at time \( t = 0 \) was connected sinusoidal current source \( i(t) = 10 \sin(1000t) \, \text{mA} \) At \( t < 0 \) the circuit was without energy.

1. Initial condition is given – \( u_c(0) = 0 \)
2. We solve integral-differential equation (in detail it will be described on 8th lecture)

\[
L \frac{d^2 i_1(t)}{dt^2} + R_2 \frac{di_1(t)}{dt} + \frac{1}{C} \int_0^t [i_1(\tau) - i(\tau)] \, d\tau - u_C(0) = 0
\]

\[
a) \quad \text{Derive}
\]

\[
L \frac{d^2 i_1(t)}{dt^2} + R_2 \frac{di_1(t)}{dt} + \frac{i_1(t) - i(t)}{C} = 0
\]

\[
b) \quad \text{Solve using method of variation of parameters}
\]

\[
\lambda^2 + \frac{R_2}{L} \lambda + \frac{1}{LC} = 0 \quad \Rightarrow \quad \lambda^2 + 1000 \lambda + 10^6 = 0
\]

\[
\lambda_{1,2} = -500 \pm \sqrt{500^2 - 10^6} = -500 \pm 866.03j
\]

\[
i_1(t) = [K_1 \cos(866.03t) + K_2 \sin(866.03t)] e^{-500t} + 0.01 \sin(1000t - \frac{\pi}{2})
\]

\[
\text{AC solved above}
\]
Laplace

\[ I_1(p) = \frac{I(p)}{p^2LC + pR_2C + 1} = \frac{0.01 \cdot 1000}{p^2 + 1000^2} - \frac{p \cdot 1 \cdot 10^{-6} + p \cdot 1000 \cdot 10^{-6} + 1}{p^2 + 1000^2} = \frac{0.01 \cdot 1000}{p^2 + 1000^2} - \frac{p \cdot 1000 + 1}{p^2 + 1000^2} = \]

\[ = 0.01 \left[ \frac{p + 500}{(p + 500)^2 + (500\sqrt{3})^2} + \frac{1}{\sqrt{3} (p + 500)^2 + (500\sqrt{3})^2} - \frac{p}{p^2 + 1000^2} \right] \]

\[ i(t) = 10 e^{-500t} \left[ \cos \left( 500 \sqrt{3} t \right) + \frac{1}{\sqrt{3}} \sin \left( 500 \sqrt{3} t \right) \right] + 10 \sin \left( 1000 t - \frac{\pi}{2} \right) \text{ mA} \]

We don’t solve any AC, steady state is part of solution
Mesh analysis:

Time domain:

\[ R_1 i_1(t) + \frac{1}{C} \int_0^t [i_1(\tau) - i_2(\tau)] \, d\tau + u_c(0+) - u(t) = 0 \]

\[ L \frac{d^2 i_2(t)}{dt^2} + R_2 i_2(t) - u_c(0+) + \frac{1}{C} \int_0^t [i_2(\tau) - i_1(\tau)] \, d\tau = 0 \]

Laplace:

\[ R_1 I_1(p) + \frac{1}{pC} [I_1(p) - I_2(p)] + \frac{u_C(0)}{p} - U(p) = 0 \]

\[ pL I_2(p) - L i_L(0) + R_2 I_2(p) - \frac{u_C(0)}{p} + \frac{1}{pC} [I_2(p) - I_1(0)] = 0 \]

Matrix notation

AC:

\[ \begin{bmatrix}
  R_1 + \frac{1}{j\omega C} & -\frac{1}{j\omega C} \\
  -\frac{1}{j\omega C} & j\omega L + R_2 + \frac{1}{j\omega C}
\end{bmatrix}
\begin{bmatrix}
  I_1(p) \\
  I_2(p)
\end{bmatrix}
= 
\begin{bmatrix}
  U(p) - \frac{u_C(0+)}{p} \\
  \frac{u_C(0+)}{p} + L i_L(0+)
\end{bmatrix} \]
**Nodal analysis:**

\[ \begin{align*}
\text{U}_1(p) & = \frac{U(p)}{R_1} + \frac{U_1(p) - U_2(p)}{pL} + pCU_1(p) - CuC(0) + \frac{i_L(0)}{p} = 0 \\
\text{U}_2(p) & = \frac{U_2(p) - U_1(p)}{pL} + \frac{U_2(p)}{R_2} - \frac{i_L(0)}{p} = 0
\end{align*} \]

\[ \begin{bmatrix}
\frac{1}{R_1} + pC + \frac{1}{pL} & \frac{1}{pL} \\
-\frac{1}{pL} & \frac{1}{pL} + \frac{1}{R_2}
\end{bmatrix} \begin{bmatrix}
\text{U}_1(p) \\
\text{U}_2(p)
\end{bmatrix} = \begin{bmatrix}
\frac{U(p)}{R_1} + CuC(0_+) - \frac{i_L(0_+)}{p} \\
\frac{i_L(0_+)}{p}
\end{bmatrix} \]

**Time domain:**

\[ \begin{align*}
u_1(t) - u(t) & = \frac{1}{R_1} \int_{0}^{t} [u_1(\tau) - u_2(\tau)] \, d\tau + C \frac{du_1(t)}{dt} + i_L(0) = 0 \\
u_2(t) & = \frac{1}{L} \int_{0}^{t} [u_2(\tau) - u_1(\tau)] \, d\tau + \frac{u_2(t)}{R_2} - i_L(0) = 0
\end{align*} \]

**Laplace:**

\[ \begin{align*}
u_1(t) - u_1(t) & = \frac{1}{R_1} \int_{0}^{t} [u_1(\tau) - u_2(\tau)] \, d\tau + C \frac{du_1(t)}{dt} + i_L(0) = 0 \\
u_2(t) & = \frac{1}{L} \int_{0}^{t} [u_2(\tau) - u_1(\tau)] \, d\tau + \frac{u_2(t)}{R_2} - i_L(0) = 0
\end{align*} \]

**Matrix notation**

**AC:**

\[ \begin{bmatrix}
\frac{1}{R_1} + j\omega C + \frac{1}{j\omega L} & -\frac{1}{j\omega L} \\
-\frac{1}{j\omega L} & \frac{1}{j\omega L} + \frac{1}{R_2}
\end{bmatrix} \begin{bmatrix}
\hat{U}_1 \\
\hat{U}_2
\end{bmatrix} = \begin{bmatrix}
\frac{\hat{U}}{R_1} \\
0
\end{bmatrix} \]

---

**Be careful – if circuit contains controlled sources, it is necessary introduce other, more complicated rules for direct writing of matrices (nullor model)!!!**

XE31EO2 - Pavel Máša - Lecture 5
It is suitable for DC, AC, Laplace

- Gaussian elimination solution, Cramer’s rule, inverse matrix – very easy with mathematical computer programs (Matlab, Maple, ...)

Could not be used in time domain